Enrollment No......................................

Faculty of Science

End Sem (Even) Examination May-2018

BC3CO15 Mathematics-IV

Programme: B.Sc.(CS) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. In a group $(G,*)$, the closure law is $\forall$ a, b $\in$ G **1**

(a) $a*b\in G$ (b) $a*b = b*a$ (c) $a+(-a)=0$ (d) $a+e=a$

ii. The identity element of the group $(\{0,1,2,3,4,5\},+_5)$ is: **1**

(a) 1 (b) 2 (c) 0 (d) 5

iii. If f is a homomorphism of (G,0) onto (G',0') where e, e' are identity elements of G and G' then: **1**

(a) f(e) = e (b) f(e) = e' (c) f(e) = 1 (d) None of these

iv. If f is an isomorphism of (G,0) onto (G',0') where e, e' are identity elements of G and G' and k is kernel of f then: **1**

(a) K = {e} (b) K = G (c) K= 0 (d) None of these

v. $(Z_n, +_n, \times_n)$ is a field, when **1**

(a) n=7 (b) n=8 (c) n=14 (d) n=16

vi. Which of the following structure is not a commutative ring with Unity? **1**

(a) (I, +, .) (b) (2I, +, .) (c) (Q, +, .) (d) (R, +, .)

vii. In order to make a subset S of a vector space V(F) to be its basis, which of the following conditions should be deleted: **1**

(a) $S \neq \emptyset$ (b) $0 \in S$

(c) S is linearly independent (d) L (S) = V.

viii. For each subspace *W* of a finite dimensional vector space, which of the following is true: **1**

(a) dimW < dimV (b) dimW $\leq$ dimV

(c) dimW > dimV (d) dimW $\geq$ dimV

ix. A linear operator T is invertible if : **1**

(a) T is one-one (b) T is onto

(c) T is one-one and onto (d) None of these

x. Eigen values of the following matrix are: **1**

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

(a) 8, 8, -1 (b) 8, 8, 1 (c) 8, -1, -1 (d) 8, 1, 1

Q.2 i. Prove that if every element of a group (G,0) is its own inverse, then (G,0) is an abelian group. **2**

ii. Any two right (left) cosets of a subgroup of a group (G,0) are either disjoint or identical. **3**

iii. State and Prove Lagrange's Theorem. **5**

OR iv. Show that the set of n-nth roots of unity form a finite abelian multiplicative group of order n. **5**

Q.3 i. The intersection of any two normal subgroups of a group (G,0) is a normal subgroup of G. **2**

ii. Let G and G' be the two groups and f : G→G' is a homomorphism of G onto G'. If K is kernel of f then prove that G / K is isomorphic to G'. **8**

OR iii. Prove that "Every finite group G is isomorphic to a permutation group". **8**

Q.4 i. Prove that a ring (R, +, ×) is without zero divisors if and only if the cancellation laws hold in R. **3**

ii. Prove that Every field is an Integral Domain but the converse is not true. **7**

OR iii. Prove that under addition and multiplication the set s = {0, 2, 4, 6, 8} (mod 10) is a ring. Does it posses unity element? **7**

Q.5 i. The necessary and sufficient condition for a non-empty subset *W* of a vector space *V(F)* to be a vector subspace of *V(F)* is: **4**

$$\forall\, a, b \in F \text{ and } \forall\, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$$

ii. If W_1 and W_2 are two vector subspaces of a finite dimensional vector space *V(F)*, then **6**

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

OR iii. State and Prove Basis theorem for Vector Spaces. **6**

Q.6 Attempt any two:

i. Let V and W be vector spaces over a field F, and let T:V→W be a linear transformation. Assuming the dimension of V is finite, then prove that: **5**

dim(V)=dim(Ker(T))+dim(Im(T)),

where dim(V) is the dimension of V, Ker is the kernel, and Im is the image.

ii. Find the range, rank, null space and nullity of the linear transformation T : V_2 (R) →V_3 (R) defined by: T(a,b) = (a + b, a – b, b) **5**

iii. Show that the matrix *A* is diagonalizable: **5**

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

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Solution ! B.Sc (CBCS) (Even) May 2018 (1)
Mathematics IV BC3C015 14 may 18

Q.1 (10)

(i) (a) $a*b \in G$

(ii) (b) (c) 0 , (d) 5

(iii) (b) $f(e) = e'$

(iv) (a) $K = \{e\}$

(v) (a) $n = 7$

(vi) (b) $(2I, +, \cdot)$

(vii) (b) $0 \in S$

(viii) (b) $\dim W \leq \dim V$

(ix) (c) T is one – one and onto

(x) (c) 8, -11, -1

Q.2 (i) Every element of (G, o) is its own inverse then (G, o) is an abelian group.

To prove $ab = ba$

Given $a^{-1} = a$ $b^{-1} = b$ $\forall a, b \in G$

Now $(ab)^{-1} = b^{-1}a^{-1}$ (By property)

But $(ab)^{-1} = ab$ (Given)

$\Rightarrow$ $b^{-1} = b$ $a^{-1} = a$ (2)

$ab = ba$

as

$G = a_1 H \cup a_2 H$ --- $\cup a_k H$ (+1) (2)

each coset has m element

Total no of element in R.H.S $= mk$

Total no of element in L.H.S $= n$

$\Rightarrow n = mk$

$\Rightarrow k = \frac{n}{m} = \frac{o(G)}{o(H)}$

$\Rightarrow \boxed{mk = n}$ (+1)

5 marks

QR To prove

$G = \{1, \omega, \omega^2 \dots \omega^{n-1}\}$ $\omega = e^{2\pi i/n}$ (1)

$\omega^n = 1$

Closure property :-

let $a, b \in G \Rightarrow a^n = 1$ $b^n = 1$

$(ab)^n = a^n b^n = 1.1$

$\Rightarrow ab$ is also n^{th} root of unity

$\Rightarrow ab \in G$

Associative :- All are complex no.

$\Rightarrow$ Associative (+1)

Identity element : $1 \in G$ $1.a = a \; \forall \; a \in G$

Inverse : inverse of 1 is 1 ~~and~~ ~~let~~ =

ω^r $1 \leq r \leq n-1$ is any element

of G, then ω^{n-r} is also $\in G$ (+1)

s.t $\omega^r . \omega^{n-r} = \omega^n = 1$

$\Rightarrow \omega^{n-r}$ is inverse of ω^r (+1)

Commutative :- multiplication of complex no. is commutative.

(+1)

Or by example $(1, i, i^2)$

5 marks

(ii) Let Ha and Hb are any two left cosets (3)
of (G, o)

And let $Ha \cap Hb \neq \emptyset$ (1)

to prove $Ha = Hb$

As $Ha \cap Hb \neq \emptyset$

$\exists\, c \in G$ s.t. $c \in Ha \cap Hb$, for $h \in H$

i.e $c = ha$ and $c = hb$

(+1)

$c \in G$ is unique element

$\Rightarrow c = ha = hb$

$ha = hb \Rightarrow \underline{Ha = Hb}$ (+1)

3 marks

(iii) The order of each subgroup of a finite group is a divisor of the order of group. (1)

Pf Let H be a subgroup of (G, o)

Let $O(H) = m$

$O(G) = n$

to prove $O(H) \mid O(G)$ i.e $\underline{n = mk}$ (+1)

$O(H) = m$, we will prove each left coset of H in G has m distinct element

$h_1, h_2, \ldots h_m$ are distinct element.

$aH = \{ah_1, ah_2, \ldots ah_m\}$

Let $ah_i = ah_j$ for $i \neq j$

$\Rightarrow h_i = h_j$ Left cancellation law

aH has m distinct element. (+1)

Let $\exists$ K dist. coset

(4)

3 (i) Let H and K be two normal sub group of G

Let $y \in H \cap H \Rightarrow y \in H$ and $y \in K$ (1)

Since H is Normal

$x \in G$ $y \in H \Rightarrow xyx^{-1} \in H$

Since K is Normal

$x \in G$ $y \in K \Rightarrow xyx^{-1} \in K$ (+1)

$\Rightarrow xyx^{-1} \in H \cap K$

$\Rightarrow H \cap K$ is Normal

2 marks

(ii) $f: G \to G'$ is homomorphism Let K be the kernel of f. to prove $G/K \cong G'$

Le $\phi: G/K \to G'$ s.t $\underline{\phi(Ka) = f(a)}$ $\forall a \in G$ (+1)

If $a, b \in G$

$Ka = Kb$

$\Rightarrow ab^{-1} \in K$

$f(ab^{-1}) = e'$

$\Rightarrow f(a) \cdot [f(b)]^{-1} = e'$

$\Rightarrow f(a)\ e' = f(b)$ (+2)

$\Rightarrow f(a) = f(b)$

$\Rightarrow \phi(Ka) = \phi Kb$ ϕ is well defined

ϕ is one

$\phi(Ka) = \phi(Kb)$ (+2)

$\Rightarrow Ka = Kb$

$\underline{\phi \text{ is onto}}$ Let $y \in G' \Rightarrow \exists a \in G'$

s.t $y = f(a)$ (+2)

$\Rightarrow Ka \in G/K$ $\phi(Ka) = f(a) = y$

for $y \in G'$ $\exists Ka \in G/K$ s.t $\phi(Ka) = f(b)$

homo $\Rightarrow \phi[(Ka)(Kb)] = \phi(Kab) = f(ab)$ (+1)

$= f(a) f(b)$

$= \phi(Ka)\phi(Kb)$ 5m

(5)

(4)(ii) Every finite group G is isomorphic to a permutation group. (Cayley's theorem)

Sol: Consider $G = \{a_1, a_2 \ldots a_n\}$ +1

$f_a : G \to G$ defined by

$f_a(x) = ax \quad \forall x \in G$

f is well defined $f_a(x) = f_a(y)$

$\Rightarrow ax = ay$

$\Rightarrow x = y$ (one-one, +2)

f is onto

if $x \in G$

$\exists\, a^{-1}x \in G$ s.t

$f_a(a^{-1}x) = a(a^{-1}x)$

$= (aa^{-1})x$

$= ex = x$ (+1)

Consider permutation

$$f_a = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ aa_1 & aa_2 & & aa_n \end{pmatrix}$$ (+1)

$G' = \{ f_a : a \in G \}$

G' is group

Closure, Associative, Identity f_e, +2

Inverse $f_{a^{-1}} \in G'$ for $f_a \in G$.

8 marks

Q.4 (i) Let $(R, +, \times)$ is without zero divisor

To prove cancellation law hold

Let $ab = ac$

$\Rightarrow a(b-c) = 0$

$\Rightarrow a = 0$ or $b - c = 0$

(without zero div)

Let $a \neq 0 \Rightarrow a^{-1} \in R$

$\Rightarrow a^{-1}a(b-c) = 0$ (1.5)

$\Rightarrow b - c = 0 \Rightarrow b = c$

$\Leftarrow$ Let cancellation law hold in R
and let $ab = 0$
$\Rightarrow$ To prove $a = 0$ or $b = 0$

Let $a \neq 0 \Rightarrow a^{-1} \in R$

$a^{-1}(ab) = a^{-1}.0 = 0$ (cancellation)

$e.b = 0 \Rightarrow \underline{\underline{b = 0}}$

or $\underline{b \neq 0 \Rightarrow a = 0}$ (1.5)

3 mark

Q 4

(ii) Let $(F, +, \cdot)$ is field

i.e. it is commutative
has unity and each non zero element
possess multiplicative inverse.

to prove it is without zero divisor
$(\Rightarrow$ it is an integral domain)

Let $ab = 0$ with $a \neq 0$ to prove $b = 0$

$a \neq 0 \Rightarrow \exists\ a^{-1} \in F$ st $a a^{-1} = a^{-1} a = e$

Now $ab = 0$

$a \neq 0 \Rightarrow a^{-1}.ab = e.b = 0$

$\Rightarrow b = 0$

If $ab = 0$
$b \neq 0 \Rightarrow abb^{-1} = 0$

$\Rightarrow a = 0$

Thus $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$

$\Rightarrow (F, +, \cdot)$ is integral domain (5)

Now $(Z, +, \cdot)$ is integral domain but non zero
ele. does not posses multiplicative inverse
$\Rightarrow$ Not a field (+2)

QR

Q4

(iii) $S = (\{0, 2, 4, 6, 8\}\ +_{10}\ \times_{10})$ is Ring.

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

(+3)

$(S, +_{10})$ is abelian group

$(S\ \times_{10})$ is semi group   Also $a \times_{10} (b +_{10} c) = a \times_{10} b +_{10} a \times_{10} b$ (+3)

$\Rightarrow (S\ +_{10}, \times_{10})$ is Ring. $\forall\ a, b \in S$

Also 6 is unit element of $\underline{\underline{S}}$ (+1)

as $2 \times_{10} 6 = 2$

$4 \times_{10} 6 = 4$

$6 \times_{10} 6 = 6$

$8 \times_{10} 6 = 8$

7 m.

5 (i) Necessary condition :- If W is subspace of V it is closed under scalar multiplication and vector addition

$\exists\ a \in F\ \ \alpha \in W \Rightarrow a\alpha \in W$

$b \in F\ \ \beta \in W \Rightarrow b\beta \in W$

$a\alpha \in W\ \ b\beta \in W \Rightarrow \underline{a\alpha + b\beta \in W}$

Sufficient condition   Take $a = 1\ \ b = 1$

$\Rightarrow \alpha \in W\ \ \ \beta \in W$

$\Rightarrow \alpha + \beta \in W$

(8)

Also since F is field $0 \in F$ and $1 \in F$
$\Rightarrow -1 \in F$

Taking $a = -1$, $b = 0$

if $\alpha \in W$

$\Rightarrow (-1)\alpha + 0\beta \in W$

$\Rightarrow -\alpha \in W$

$\Rightarrow$ additive inverse exist

Taking $a = 0$ $b = 0$) we get

$0\alpha + 0\beta = 0 \in W$

Taking $\beta = 0$, we ~~get~~

$a, b \in F$ $\alpha \in W$

$\Rightarrow a\alpha \in W$ closed under +, scalar multiplication

Remaining properties hold in W as in V

$\Rightarrow W$ is vector subspace of $V(F)$

4 marks

(ii) $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Let $\dim(W_1 \cap W_2) = k$

$S = (v_1, v_2 \cdots v_k)$ be a basis of $W_1 \cap W_2$ (+1)

$S \subset W_1$, $S \subseteq W_2$

$S \subset W_1$ L.I can be extended to form basis

Let $\{v_1, v_2 \quad v_k, \alpha_1 \alpha_2 \cdots \alpha_m\}$ be a basis of W_1 (+1)

$\dim W_1 = k + m$

Let $S \in W_2$ $(v_1, v_2 \cdots v_k, \gamma_1, \gamma_2 \quad \gamma_l)$ be a basis of W_2 (+1)

$\dim W_2 = k + m$

To prove $\dim(W_1 + W_2) = k + l + m$ (+1)

Show $S_1 = \{ v_1, v_2 -- v_k, \alpha_1, \alpha_2 -- \alpha_m, \beta_1, \beta_2 -- \beta_l \}$ (9)
is L.I. (+1)

Also $L(S_1) \subseteq W_1 + W_2$ (+1)

$W_1 + W_2 \subseteq L(S_1)$

$\Rightarrow \dim(W_1 + W_2) = k + l + m$ Then the Th

6 mark

iii) Basis theorem

There exist a basis for each finite dimensional vector space.

Pf Let $V(F)$ be vector space generated by $S = \{\alpha_1, \alpha_2 -- \alpha_n\}$ of V +1

If S is L.I, S is basis

If S is L.D $\exists$ α_k $2 \le k \le n$ in S

for $\alpha_k = a_1\alpha_1 + a_2\alpha_2 - + a_{k-1}\alpha_{k-1}$ (+2)

Examine α_k then $\{\alpha_1, \alpha_2 -- \alpha_{k-1}, \alpha_{k+1}, -- \alpha_n\}$ is L.D

the remaining set S_2 (+2)

if S_2 is L.I it is basis otherwise

done in previous way, finally we will

get a basis for V (+1)

6 marks

6 (i) Rank Nullity theorem

$T: V \to V$

To prove

$\dim(V) = \dim(\ker T) + \dim(\text{Ran } T)$

Pf Let $\dim(V) = n$

$N(T)$ is a $\ker(T)$

$N(T)$ is finite dimensional

Let $B_1 = \{\alpha_1, \alpha_2 -- \alpha_k\}$ be a basis of $N(T)$

$k \leq n$

V is finite dim B_1 is L.I. can be extended

to form basis of V

let $B_2 = \{\alpha_1, \alpha_2 -- \alpha_k, \alpha_{k+1} -- \alpha_n\}$

be a basis of V.

Then $T(\alpha_1)\ T(\alpha_2) -- T(\alpha_k) -- T(\alpha_n) \in R(T)$

Now $\{T(\alpha_{k+1}), T(\alpha_{k+2}) -- T(\alpha_n)\}$

form a basis of $R(T)$ (To prove,

$\beta \in R(T)\ \exists\ \alpha \in V$ s.t

$T(\alpha) = \beta$

$\alpha \in V \Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + a_k\alpha_k$

$+ a_{k+1}\alpha_{k+1} -- + a_n\alpha_n$

$\alpha = a_{k+1}\alpha_{k+1} + -- a_n\alpha_n$

as $\alpha_1 -- \alpha_k \in B_1 \in N(T)$

$T(\alpha) = a_{k+1}T(\alpha_{k+1}) -- + a_n T(\alpha_n)$

$L\{B_3\} = R(T)$

Now $T(a_{k+1}\alpha_{k+1} + a_{k+2}\alpha_{k+2} -- + a_n\alpha_n) = 0$

also $a_{k+1}T(\alpha_{k+1}) -- + a_n T(\alpha_{k+1}) = 0$

$\Rightarrow \alpha_{k+1}, a_{k+1}. -- a_n\alpha_n \in N(T)$

$\Rightarrow a_1\alpha_1 -- a_k\alpha_k - a_{k+1}\alpha_{k+1} -- -a_n\alpha_n = 0$

$\Rightarrow a_1 = a_2 -- = a_n = 0$ as $(\alpha_1 -- \alpha_n)$ bas

$\Rightarrow B_3$ for basis of $R(T)$ (1)

$\dim(R(T)) = n - k = \text{Rank}(T)$

$\Rightarrow \dim(V) = \dim(N(T)) + \dim(R(T))$

(ii) $T(a,b) = (a+b, a-b, b)$ $T: V_2(R) \to V_3(R)$

is linear transformation

$N(T) = \{ \alpha \in V_2(R) \; ; \; T(\alpha) = (0,0,0) \in V_3 R \}$

$T(\alpha) = T(a,b) = (a+b, a-b, b)$

$= (0, 0, 0)$

$\Rightarrow a+b = 0$

$a-b = 0$

$b = 0$

$\Rightarrow a = 0, \; b = 0$

$\Rightarrow (0,0) \in N(T)$

$N(T)$ is zero subspace of $V_2(R)$

$\dim(N(T)) = 0$ (2.5)

Range of T: $\{(1,0) \; (0,1)\}$ is basis of $V_2(R)$

$T(1,0) = (1, 1, 0)$

$T(0,1) = (1, -1, 1)$

Now $(1,1,0)$ and $(1,-1,1)$ are LI

as $a(1,1,0) + b(1,-1,1) = 0$

$\Rightarrow a = 0 \quad b = 0$

Then $\dim(R(T)) = 2$

$R(T) = L\{(1,1,0) \; (1,-1,1)\}$ (2.5)

(12)

(Q5) (iii) $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Eigen values $\underline{\underline{1, 5, 5}}$ (+1)

Eigen vector corresponding to $\lambda = 1$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (+1)

Eigen vector corresponding to λ_2

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (+2)

Also
Nullity of $(A - 5I) = 2$
$\Rightarrow$ A is diagonalizable

With $P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (+1)

5 marks